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COMPLETENESS CRITERIA FOR SEQUENCES OF SPECIAL FUNCTIONS

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Abstract: Two L^p completeness criteria for sequences of functions written in the form $f(\lambda_n x)$, where λ_n is the nth zero of another function g, are obtained. The results are illustrated with applications to sequences of special functions.

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1. Introduction

A sequence of functions $\{f_n\}$ is complete in $L^p[\mu, X]$ provided the relations

$$\int_X y f_n d\mu = 0$$

for $n = 1, 2, ...$, with $y \in L^q[\mu, X]$ and $1/p + 1/q = 1$, imply $y = 0$ almost everywhere. This is equivalent to say that $\{f_n\}$ spans the whole space $L^p[\mu, X]$. If X is a finite interval, then completeness in $L^1[\mu, X]$ carries with it completeness in $L^p[\mu, X]$, $p > 1$. We will say that a set is complete $L[\mu, X]$ if it is complete in $L^1[\mu, X]$. This note deals with completeness properties of sequences $\{f_n\}$ of functions defined by

$$f_n(x) = f(\lambda_n x) \tag{1.1}$$

where f is an entire function and λ_n is the nth element of a sequence of real numbers.

The first significant paper on completeness of nonorthogonal sequences of the type (1.1) was Paley and Wiener´s [11], where $f(x) = e^{ix}$ was considered. It was the groundwork for the branch of analysis called *nonharmonic fourier series* [9]. Boas and Pollard studied in [4] sequences of Fourier-Bessel functions $\{J_\nu(\lambda_n x)\}$ where λ_n is not necessarily the nth zero of J_ν. In [3], Boas gave necessary and sufficient conditions for the completeness of $\{f(\lambda_n x)\}$ in Hardy spaces. A survey on completeness of sets of special functions can

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be found in [7]. The recent interest in $q-$Fourier series that followed their introduction in [5] brought with it natural questions about completeness of sets of basic hypergeometric series [1], [2], [8], [13].

The results obtained in our discussion will assure, under certain conditions, the completeness of the sequence $\{f(\lambda_n x)\}$, when λ_n is the nth zero of g, where g is a suitable chosen entire function. Two criteria will be obtained, one valid for functions of order less than one and another valid for functions of order less than two. The proofs use classical entire function theory, namely arguments of the Phragmén-Lindelöf type very common in this situations (see, for instance, [10], [2]). The results are stated in a degree of generalization sufficient to cover both the cases corresponding to the usual dx measure in the real line and to the discrete $d_q x$ measure associated to Jackson´s $q-$integral. This will be illustrated in the last section of the paper with the selection of some examples of special functions.

2. Completeness criteria

Some facts from the classical entire function theory will be used. The maximum modulus of the entire function f is defined as

$$M(r;f) = \max_{|z|=r} |f(z)|$$

and the order of f as

$$\varrho(f) = \lim_{r\to\infty} \frac{\log\log M(r;f)}{\log r} \tag{2.1}$$

In the case where f is a canonical product with zeros $r_1, r_2, \ldots$, the order of f is equal to the greatest lower bound of all the τ for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|r_n|^{\tau}}$$

converges. From this it is easy to verify that if $A \subset B$ then

$$\varrho\left[\prod_{n\in A}^{\infty}\left(1-\frac{z}{r_n}\right)\right] \leq \varrho\left[\prod_{n\in B}^{\infty}\left(1-\frac{z}{r_n}\right)\right] \tag{2.2}$$

The proof of the main result requires the following form of the Phragmén-Lindelöf Principle [9].

If the order of an entire function f is less than σ and f is bounded on the limiting rays of an angle with opening π/σ then f is bounded on the region defined by the rays .

Now consider the general setting to be used in the two theorems in this section: Two entire functions f and g defined by the power series expansions

$$f(z) = \sum_{n=0}^{\infty}(-1)^n a_n z^{2n} \tag{2.3}$$

and

$$g(z) = \sum_{n=0}^{\infty}(-1)^n b_n z^{2n} \tag{2.4}$$

Denote by ζ_n the nth zero of f and by λ_n the nth zero of g. Our first result is the following

Theorem 1. *Let μ be a positive defined measure. If the order of f is less than one, then the sequence $\{f(\lambda_n x)\}$ is complete $L[\mu,(0,1)]$ if, as $n \to \infty$,*

$$\frac{a_n}{b_n} \to 0 \tag{2.5}$$

Proof: Let $y \in L^{\infty}[\mu,(0,1)]$ such that for $n = 1, 2, ...$

$$\int_0^1 y(x)\, f(\lambda_n x) d\mu(x) = 0 \tag{2.6}$$

and set

$$h(w) = \frac{H(w)}{g(w)} \tag{2.7}$$

where

$$H(w) = \int_0^1 y(x)\, f(wx)\, d\mu(x) \tag{2.8}$$

The idea of the proof is to show that h is constant and conclude from it that y must be null almost everywhere. This will be carried out in three steps.

Step 1: *The function h is entire and $\varrho(h) \leq 1$.*

Because of its continuity, f is bounded on every disk of the complex plane. Therefore, the maximum of f on a disk of radius r exists and the inequality

$$M(r;H) \leq M(r;f) \left| \int_0^1 y(x) d\mu(x) \right| \tag{2.9}$$

holds. From this we infer that the integral defining H converges uniformly in compact sets. The condition (2.6) forces every zero of g to be a zero of H and the identity (2.7) shows that h is an entire function with less zeros than H; by (2.2), the order of h is less or equal to the order of H. On the other side, the order of H is less or equal to the order of f. This becomes clear using (2.1) and inequality (2.9). It follows $\rho(h) \leq \rho(f) < 1$.

Step 2: *The function h is constant.*

Condition (2.5) implies the existence of a constant $A > 0$ such that $a_n \leq Ab_n$. Then $|x| \leq 1$ gives

$$f(itx) = \sum_{n=0}^{\infty} a_n t^{2n} x^{2n} \leq \sum_{n=0}^{\infty} Ab_n t^{2n} x^{2n} \leq A \sum_{n=0}^{\infty} b_n t^{2n} = Ag(it)$$

Taking into account that μ is a positive-defined measure, this inequality allows to estimate the integral in (2.7)

$$\left| \int_0^1 y(x) f(itx) d\mu(x) \right| \leq A \left| g(it) \right| \int_0^1 \left| y(x) \right| d\mu(x)$$

or equivalently

$$\left| h(it) \right| \leq A \int_0^1 \left| y(x) \right| d\mu(x)$$

That is, h is bounded on the imaginary axis. By Step1, $\varrho(h) < 1$. The Phragmén-Lindelöf theorem with $\sigma = 1$ shows that h is bounded in the complex plane. By Liounville´s theorem h is a constant.

Step 3: *The function y is null almost everywhere.*

Step 2 shows the existence of a constant C such that $h(w) = C$ for every w in the complex plane. Rewrite this as

$$\int_0^1 y(x) f(wx) d\mu(x) - g(w) C = 0$$

Using the series expansion for $f(wx)$ and $g(w)$, the identity theorem for analytical functions gives

$$\frac{a_n}{b_n} \int_0^1 y(x) x^{2n} d\mu(x) = C \tag{2.10}$$

On the other side, $x < 1$ implies

$$\left|\frac{a_n}{b_n}\int_0^1 y(x)x^{2n}d\mu(x)\right| \leq \frac{a_n}{b_n}\left|\int_0^1 y(x)d\mu(x)\right| \tag{2.11}$$

Taking the limit when $n \to \infty$, (2.5) and (2.10) show that C is null. As a result, for $n = 1, 2, ...$,

$$\int_0^1 y(x)x^{2n}d\mu(x) = 0$$

Finally, the completeness of x^{2n} in $L[\mu,(0,1)]$ (by the Müntz-Szász theorem) shows that $y = 0$ almost everywhere. ■

Now this result will be extended to the bigger class of entire function of order less than two. However, this will require a restriction on the behavior of the zeros. With the same notational setting of the preceding theorem, the following holds:

Theorem 2. *If the order of f is less than two, then the sequence $\{f(\lambda_n x)\}$ is complete $L[\mu,(0,1)]$ if, together with (2.5), the following condition holds*

$$\lambda_n \leq \zeta_n$$

Proof: Consider h defined as in (2.7). The proof goes by the lines of the proof of Theorem 1. Only Step 2 requires a modification because now $\varrho(h) < 2$. The way to compensate this is to make the estimates along smaller regions of the complex plane. Consider the angles defined by the lines $\arg z = \pm\frac{\pi}{4}$ and $\arg z = \pm\frac{3\pi}{4}$. These lines are the bounds of an angle of opening $\frac{\pi}{2}$. If z belongs to one of the lines, then z^2 belongs to the imaginary axis. Say $z^2 = it$, $t \in \mathbf{R}$. Now, by the Hadamard factorization theorem, the infinite product expansion holds

$$\left|\frac{f(zx)}{g(z)}\right| = \prod_{n=1}^{\infty}\left|\frac{\left(1-\frac{itx^2}{\zeta_n^2}\right)}{\left(1-\frac{it}{\lambda_n^2}\right)}\right| = \prod_{n=1}^{\infty}\left[\frac{1+\frac{t^2x^4}{\zeta_n^4}}{1+\frac{t^2}{\lambda_n^4}}\right]^{\frac{1}{2}}$$

and the hypothesis $\lambda_n \leq \zeta_n$ together with $x \leq 1$ implies

$$\frac{1+\frac{t^2x^4}{\zeta_n^4}}{1+\frac{t^2}{\lambda_n^4}} \leq 1$$

Now, clearly

$$|f(zx)| \leq |g(z)|$$

From this we infer that the function h is bounded on the sides of an angle of opening $\frac{\pi}{2}$. Applying the Phragmén-Lindelöf theorem with $\sigma = 2$ it follows that h is bounded in the complex plane and, as before, it is a constant. ■

3. Complete sets of special functions

Before considering some special cases of the Theorems in the preceding section, it is convenient to recall that if a function is given in its series form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then the order $\varrho(f)$ is given by

$$\varrho(f) = \lim_{n\to\infty} \sup \frac{n \log n}{\log(1/|a_n|)} \tag{3.1}$$

3.1. Sets of Bessel functions. The first example is an application of Theorem 2 to the classical Bessel function. The Bessel function of order $\nu > -1$ is defined by the power series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}$$

It is an entire function of order one.

Theorem 3. *Let $\alpha, \nu > 0$ such that $\alpha < \nu$. The sequence $\{J_\nu(xj_{n\alpha})\}$ is then complete $L(0,1)$*

Proof: Consider $f(z) = (z/2)^{-\nu} J_\nu(z)$ and $g(z) = (z/2)^{-u} J_\alpha(z)$. Both f and g are entire functions of the form considered in Theorem 2, with

$$a_n = 1/2^{2n} n!\Gamma(\nu+n+1)$$

$$b_n = 1/2^{2n} n!\Gamma(\alpha+n+1)$$

The identity $\Gamma(x+n) = \Gamma(x)(x)_{n+1}$ implies

$$\frac{a_n}{b_n} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\nu+n+1)} = \frac{\Gamma(\alpha)\,(\alpha)_{n+1}}{\Gamma(\nu)\,(\nu)_{n+1}} \to 0 as n \to \infty$$

Furthermore, it is a well known fact from the theory of Bessel functions [12, pag. 508] that if $\alpha < \nu$ then $j_{n\alpha} < j_{n\nu}$ for all n. Consequently, Theorem 2 apply. ■

3.2. Sets of $q-$special functions. Theorem 1 is very convenient to be applied to sets of $q-$special functions. Many often these functions are of order zero, corresponding to the situation where there is no restriction on the behavior of the zeros. In this section, we will follow the notations in [6]. Consider $0 < q < 1$, and define the q-shifted factorial for n finite and different from zero as

$$(a;q)_n = (1-q)(1-aq)\ldots\left(1-aq^{n-1}\right)$$

and the zero and infinite cases as

$$(a;q)_0 = 1$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$$

The $q-$difference operator D_q is

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x} \tag{3.2}$$

and the $q-$integral in the interval $(0,1)$

$$\int_0^1 f(t)\,d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)\,q^n \tag{3.3}$$

These two definitions appear in the formula of $q-$ integration by parts :

$$\int_0^1 G(qx)\left[D_q f(x)\right]d_qx = f(1)G(1) - f(0)G(0) - \int_0^1 f(x)D_qG(x)d_qx \tag{3.4}$$

Observe that the $q-$integral (3.3) is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points q^k, with the jump at the point q^k being q^k. We will denote by $L_q(0,1)$ the Banach space induced by the norm

$$\|f\| = \int_0^1 |f(t)|\,d_qt$$

There are three $q-$analogues of the Bessel function due to F. H. Jackson and denoted by $J_\nu^{(1)}(z;q)$ $J_\nu^{(2)}(z;q)$ and $J_\nu^{(3)}(z;q)$. A well known formula by Hahn displays $J_\nu^{(2)}(z;q)$ as an analytical continuation of $J_\nu^{(1)}(z;q)$. Therefore, just the second and the third $q-$analogues are considered. Their definition, in series form is

$$J_{\nu}^{(2)}(z;q)=\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\sum_{n=0}^{\infty}(-1)^{n}\frac{q^{n(\nu+1)}}{(q^{\nu+1};q)_{n}(q;q)_{n}}z^{2n+\nu} \tag{3.5}$$

$$J_{\nu}^{(3)}(z;q)=\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\sum_{n=0}^{\infty}(-1)^{n}\frac{q^{n(n+1)/2}}{(q^{\nu+1};q)_{n}(q;q)_{n}}z^{2n+\nu} \tag{3.6}$$

The Euler formula for the series form of an infinite product will be critical on the remainder.

$$(z;q)_{\infty}=\sum_{n=0}^{\infty}(-1)^{n}\frac{q^{n(n-1)/2}}{(q;q)_{n}}x^{n} \tag{3.7}$$

In the context of Theorem 1, consider the measure $d\mu(t)$ to be the $d_{q}t$ from (3.3).

Theorem 4. *If $\nu>-1$, the sequence $\left\{J_{\nu}^{(3)}\left(qx\lambda_{n};q^{2}\right)\right\}$ is complete $L_{q}(0,1)$ in each of the following two situations:*

(a) $\lambda_{n}=j_{n,\alpha}^{(3)}$, where $j_{n,\alpha}^{(3)}$ is the *nth* zero of the function $J_{\alpha}^{(3)}\left(x;q^{2}\right)$ and $\alpha>-1$.
(b) $\lambda_{n}=q^{-n}$

Proof: To prove (a) consider f and g defined as

$$f(z)=z^{-\nu}(q^{2};q^{2})_{\infty}/(q^{2\nu+2};q^{2})_{\infty}J_{\nu}^{(3)}\left(z;q^{2}\right)$$

$$g(z)=z^{-\alpha}(q^{2};q^{2})_{\infty}/(q^{2\alpha+2};q^{2})_{\infty}J_{\alpha}^{(3)}(z;q^{2})$$

Both f and g are functions of order 0. Consequently, Theorem 1 holds with

$$a_{n}=q^{n(n+1)+4n}/(q^{2u+2};q^{2})_{n}(q^{2};q^{2})_{n}$$

$$b_{n}=q^{n(n+1)}/(q^{2\alpha+2};q^{2})_{n}(q^{2};q^{2})_{n}$$

and clearly

$$\lim_{n\to\infty}\frac{a_{n}}{b_{n}}=0$$

To prove (b) choose f as in (a) and $g(z)=(z^{2};q^{2})_{\infty}$. Expand g by means of the series representation (3.7). The result follows in a straightforward manner from Theorem 1. ∎

The case (a) of the preceding theorem was already proved in [2]. A similar result can be stated with the $J_\nu^{(3)}(z;q)$ replaced by a $J_\nu^{(2)}(z;q)$, but before applying Theorem 1 a preliminary Lemma is required.

Lemma 1. *Let λ_n define a sequence of real numbers. For every $\nu > -1$, if the sequence $\{x^{-\nu-1}J_{\nu+1}^{(2)}(q\lambda_n x;q^2)\}$ is complete $L_q(0,1)$ then the sequence $\{x^{-\nu}J_\nu^{(2)}(q\lambda_n x;q^2)\}$ is also complete $L_q(0,1)$*

Proof: Let $y(x) \in L_q(0,1)$ such that for every $n = 1, 2...$

$$\int_0^1 y(x)x^{-\nu}J_\nu^{(2)}(\lambda_n qx;q^2)dqx = 0 \tag{3.8}$$

The $q-$difference operator (3.2) acting on the power series (3.5) gives

$$D_q\left[x^{-\nu}J_\nu^{(2)}(\lambda_n x;q^2)\right] = -\lambda_n x^{-\nu}q^{\nu+1}J_{\nu+1}^{(2)}(\lambda_n xq;q^2) \tag{3.9}$$

Now, use the $q-$integration by parts formula (3.4) and (3.9) to obtain the identity

$$\int_0^1 y(x)x^{-\nu}J_\nu^{(2)}(\lambda_n qx;q^2)d_qx =$$

$$q^{\nu+1}\lambda_n\int_0^1 x^{-\nu-1}J_{\nu+1}^{(2)}(q\lambda_n x;q^2)\left[x\int_0^x (q\lambda_n t)^\nu y(t)d_qt\right]d_qt \tag{3.10}$$

By (3.8), the expression (3.10) is zero for every $n = 1, 2, ...$Under the hypothesis, $\{x^{-\nu-1}J_{\nu+1}^{(2)}(q\lambda_n x;q^2)\}$ is complete in $L_q(0,1)$. Clearly $x\int_0^x y(t)d_qt \in L_q(0,1)$ and thus, for $m = 1, 2, ..$

$$\int_0^{q^m} y(t)d_qt = 0 \tag{3.11}$$

This implies $y(q^m) = 0$ for every $m = 1, 2, ..$ ∎

Theorem 5. *If $\nu > -1$, the sequence $J_\nu^{(2)}\left(qx\lambda_n;q^2\right)$ is complete $L_q(0,1)$ in each of the following situations:*

(a) $\lambda_n = j_{n\alpha}^{(2)}$, where $j_{n\alpha}^{(2)}$ is the *nth* zero of the function $J_\alpha^{(2)}\left(x;q^2\right)$ and $\alpha > -1$.
(b) $\lambda_n = q^{-n/2}$

Proof: An application of Theorem 1 establishes (a) when $\nu < \alpha+2$. Iteration of Lemma 1 yields the result when $\alpha > -1$. On the other side, (b) follows directly from Theorem 1 choosing $g(z) = (z^2;q^4)_\infty$. ∎

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